Arithmetic Dynamics REU

Dynamics is the study of iteration of functions. In particular, *rational dynamics* is the study of iteration of rational functions, like $f(x) = x + x^{-1}$ or $g(x) = x^2 - 3$. When you iterate $f(x)$, you are studying the sequence of functions gotten by composing f with itself repeatedly, like $f \circ f = f(f(x))$, $f \circ f \circ f = f(f(f(x)))$, and so on, and considering properties of the collection of iterates.

The colorful pictures of fractals one often sees, like Julia sets and the Mandelbrot set, are graphical representations of phenomena in rational dynamics: they concern questions of *stability*: if two points x and y start out close together, do their iterates always remain close, or do they diverge apart? If the coefficients of $f(x)$ are perturbed slightly to get a new function $\tilde{f}(x)$, do the iterates of $f(x)$ and $\tilde{f}(x)$ behave similarly, or differently?

Other questions studied in dynamics concern periodic and preperiodic points. A point is *periodic* for $f(x)$ if some iterate of f brings it back to to itself. For example, if $f(x) = x^2 - 1$, then 0 is periodic of order two, because $f(0) = -1$ and $f(-1) = 0$. A point is called *pre-periodic* if under iteration it eventually falls into a cycle: for example, when $f(x) = x^2 - 1$, then 1 is preperiodic because $f(1) = 0$, and $f(0) = -1$, $f(-1) = 0$.

The field of *arithmetic dynamics* asks number-theoretic questions about functions with rational or integer coefficients. It is a theorem that a rational function $f(x)$ with coefficients in $\mathbb{Q}$ can have at most finitely many preperiodic points which belong to $\mathbb{Q}$. An important open question strengthening this is the Morton-Silverman *Uniform Boundedness Conjecture*:

> For each $d \geq 2$, there is a uniform bound $C(d)$ such that no rational function $f(x) \in \mathbb{Q}(x)$ of degree d can have more than $C(d)$ preperiodic points in $\mathbb{Q}$.

Similar questions can be asked for restricted classes of functions, like polynomials. It is thought that for quadratic polynomials in $\mathbb{Q}[x]$, the bound is 8.

One way of approaching questions for dynamics of functions $f(x) \in \mathbb{Q}(x)$ is to study them over larger fields like the real numbers $\mathbb{R}$, or the p-adic field $\mathbb{Q}_p$ for a prime number p. You may not have encountered p-adic numbers yet. To explain them, recall that the field $\mathbb{R}$ consists of numbers with decimal expansions which extend "infinitely far to the right". The field $\mathbb{Q}_p$ consist of "numbers" with base p expansions which extend "infinitely far to the left". For instance when $p = 2$, we are allowed only to use the digits 0 and 1, and the 2-adic number $\alpha = \ldots 1111111_2$ equals -1, because when we consider the sum $\alpha + 1$, the "carries" under addition lead to the infinite string $0 = \ldots 00000$. The p-adic fields give a sophisticated way of considering limits of systems of congruences $\pmod{p}$, $\pmod{p^2}$, $\ldots$, $\pmod{p^n}$ for all n.

Dr. Rumely's current research concerns dynamics of functions in $\mathbb{Q}_p(x)$. For example, he has recently shown that there is a new invariant of a p-adic rational function, called its *crucial set*. When $f(x)$ has degree d, its crucial set consists of a weighted sum of points with total weight $d - 1$. For a given d, there are only finitely many possible configurations the crucial set can have, and the nature of the crucial set is expected to exert strong control over the dynamics of the function.

This REU will begin with a short course on dynamics, and then will move on to research. The exact topics for research haven't been decided yet; two possibilities include investigating the Uniform Boundedness Conjecture numerically for classes of functions that haven't been studied before, or classifying the possible configurations of the crucial set. In any case, the goal of the REU will be for students to experience the process of mathematical research on open problems.