MATH 8400: Algebraic Number Theory I

Course description

This course is an introduction to arithmetic beyond **Z**.

To give you just the briefest idea of what this means, you have surely heard of the *fundamental theorem of arithmetic*: every integer not 0 or ±1 can be factored uniquely as a product of primes. You probably learned in your first algebra course that the analogous result holds if we adjoin a square root of -1 to **Z**, but not if we adjoin a square root of -5.

Less well known is the following: Despite the failure of unique factorization in **Z**[$\sqrt{-5}$], any two factorizations of the same element of **Z**[$\sqrt{-5}$] have the same length. In other words, while the factorizations themselves can be different, the number of irreducibles involved cannot.[1]

It turns out that all of these phenomena concerning elementwise factorization can be explained by looking at factorizations of ideals. We will spend a large portion of the course developing this theory, which was pioneered by Dedekind and Kummer in the latter half of the 19th century. The scope of this theory extends far beyond **Z**[$\sqrt{-5}$]; a satisfactory theory of factorization of ideals into prime ideals can be developed in the "ring of integers" of any "number field". (We will define the terms in quotes on the first day of class.)

The tools you will be introduced to in this course are part of the standard tool chest of every working number theorist. As such, this course is **strongly recommended** for anyone pursuing further studies in number theory.

We will aim to cover the standard core topics in a first course in algebraic number theory:

- Basic concepts: Norms, traces, discriminants
- Dedekind domains
- Factoring of prime ideals in extensions
- Finiteness of the ideal class group
- Dirichlet's theorem on the structure of the unit group
- Special examples: Quadratic and cyclotomic fields
- Factoring of primes in Galois extensions
- Statement of algebro-analytic results (including Chebotarev's density theorem)
- Other topics as time permits

---

[1] *Example*: The canonical example of a failure of unique factorization in this ring is $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$; in this case, both factorizations of 6 have length 2.